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Level repulsion near integrability: a random matrix analogy

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Abstract. Using the analogy between the statistics of the levels of quantum Hamiltonians and the eigenvalues of random matrices we use (an appropriate choice of) the latter in order to study the transition region near integrability. We show that the nearest-neighbour spacing distribution is linear for small spacings while the inverse of its slope is proportional to the amplitude of the (integrability-destroying) perturbation.

The understanding of quantum chaos has considerably progressed thanks to the study of autonomous Hamiltonian systems [1]. Semiclassical arguments [2] and accurate numerical calculations [3] have allowed the formulation of conjectures relating the statistics of the energy levels of ergodic systems to the statistics of eigenvalues of random matrices (RM) [4], which are well known. In fact Dyson [5] has shown that, depending on the symmetries of the system, the statistics fall into one of three generic classes. For Hamiltonians with time-reversal invariance (and integer-spin particles) the distribution of the spacings between nearest-neighbours (NNSD) is very close to the famous Wigner surmise $(\pi/2)s\ \mathrm{e}^{-\pi/4s^2}$ [6]. When time-reversal invariance is broken [7], or half-integer-spin particles (in the absence of spatial symmetries), are considered [8], the repulsion between levels becomes stronger, in agreement with the RM predictions. However the similarity of the level statistics of RM and quantum Hamiltonians cannot be due to a presumed random character of the matrix elements of the latter (which was shown in [9] not to be present). Furthermore when one looks at fine-scale level fluctuations [10] it turns out that the RM theory does not suffice in order to reproduce all the details. In fact, the existence of classical periodic orbits, as shown by Berry [11], leads to deviations between the quantal behaviour of ergodic systems and simple RM predictions. Still if one limits oneself to the study of quantities such as the NNSD the predictions of the RM theory can be extremely accurate. Moreover this agreement shows that the behaviour of ergodic systems is universal, i.e. the statistics do not depend on the fine details of the system.

At the antipodes of ergodicity lies complete integrability. In this case also the statistics of the levels of the system are universal. For the NNSD, for example, one obtains a Poisson distribution, e^{-s}, a result that can be derived from very general semiclassical considerations [12] and which has been amply materialized in realistic calculations. (Still the fluctuations around the Poisson distribution, in numerically obtained spectra, can be quite large [13], depending on the system one studies. This makes the treatment of integrable systems particularly delicate.) The Poisson statistics is precisely what results from a collection of levels, (given by the elements of a diagonal matrix) randomly distributed in a given interval. Thus again in the integrable case one finds universality and a relation with the statistics of random matrices' eigenvalues.

Naturally the question arises as to what happens in the transition region between integrability and ergodicity. The main bulk of physical systems lies precisely in this intermediate region and the mixing problem is still open even on the classical level [14]. Several studies [15 and references therein] have been devoted to the analysis of this problem and it is clear that no universal description of the entire spectral statistics can be expected in this region. Roughly speaking, if one considers a Hamiltonian whose classical behaviour can be made to change from integrable to ergodic, for example through the variation of a parameter, the NNSD moves from a Poisson to a Wigner-type distribution. Several prescriptions have been given for the description of the NNSD in the intermediate region. The most popular ones are the ones due to Berry and Robnik [16] and to Brody [17]. Curiously enough, both present some flaws. The Berry–Robnik distribution reads:

$$P_{\mathrm{BR}}(s)=\mathrm{e}^{(q-1)s}\{(1-q)^2\,\mathrm{erfc}(\sqrt{\pi}qs/2)+[2q(1-q)+(\pi/2)q^3s]\,\mathrm{e}^{-(\pi/4)q^2s^2}\} \tag{1}$$

where q is the fraction of the phase-space filled with chaotic orbits. While its limits at $q=0$ and $q=1$ are correct, for $0<q<1$, $P_{\mathrm{BR}}(0)$ has non-zero value. This is unrealistic as we know that any small perturbation of degenerate levels introduces a repulsion, at least at small range, and thus lifts degeneracies, i.e. $P(0)=0$. Of course if one looks at numerically obtained NNSD with a large bin size, the agreement with the Berry-Robnik distribution is fair enough. However detailed calculations show that at small splittings the agreement breaks down [18]. The Brody distribution:

$$P_{\mathrm{B}}(s)=\alpha(q+1)s^q\,\mathrm{e}^{-\alpha s^{q+1}} \qquad \text{with } \alpha=\{\Gamma[(q+2)/(q+1)]\}^{q+1} \tag{2}$$

on the other hand vanishes as $s\to 0$ but has an infinite derivative at that point, an unrealistic feature. Due to the vanishing at $s=0$, the Brody distribution overshoots the Poisson exponential at larger s. (This fact may be at the origin of the success of the Brody distribution in representing numerically obtained NNSD).

With the two most popular parametrizations of the NNSD in the transition region presenting some undesirable feature we are left with no clear guide. The aim of this paper is to analyse the level statistics in the region near integrability in order to extract any possible universal behaviour, at least for small spacings. The one thing that seems reasonable to assume is that the NNSD vanishes at $s=0$ for all but the strictly integrable systems. Indeed Robnik [19] obtained this result using perturbation theory of pairs of levels. Here we present a slightly different 2×2 random matrix model (see also [20]). Let us start with a matrix:

$$\begin{pmatrix} \varepsilon-a/2 & b \\ b & \varepsilon+a/2 \end{pmatrix}$$

where a and b are randomly chosen according to the following laws:

$$p(a>A)=\mathrm{e}^{-A}$$

(where $p(a>A)$ means the probability for a to be larger than A) and

$$p(b>B)=F(B)$$

with $F(0)=1$, $F(\infty)=0$. Thus in the absence of the off-diagonal perturbation b, the level spacing is distributed according to a Poisson law. When b is present the splitting becomes $\delta = (a^2 + 4b^2)^{1/2}$ and the probability $p(\delta > s)$ is just:

$$p(\delta > s) = e^{-s} + \int_0^s dA\, F(\sqrt{s^2 - A^2}/2)\, e^{-A}. \tag{3}$$

Thus the distribution of level spacings is:

$$P(s) \equiv -\frac{dp}{ds} = \int_0^s dA \frac{s}{2\sqrt{s^2 - A^2}} f(\sqrt{s^2 - A^2}/2)\, e^{-A} \tag{4}$$

where $f(u) = -dF/du$. Putting $A = s \sin\theta$ we get:

$$P(s) = \frac{s}{2} \int_0^{\pi/2} d\theta\, f\left(\frac{s}{2}\cos\theta\right) e^{-s\sin\theta}. \tag{5}$$

We remark that if $f(0)$ is finite and non-zero then $P(s) \underset{s\to 0}{\rightarrow} s\pi f(0)/4$, i.e. the spacing distribution vanishes linearly at zero spacing. (In fact, it would still vanish, though not linearly, if $f(s)$ were divergent at $s = 0$ but integrable over $[0, 1]$.) Next we introduce a family of distributions F through the scaling $F(u) = \Phi(u/\lambda)$ and define $\phi = -\Phi'$. We obtain thus:

$$P(s) = \frac{s}{2\lambda} \int_0^{\pi/2} d\theta\, \phi(s\cos\theta/2\lambda)\, e^{-s\sin\theta} \tag{6}$$

Thus the use of random matrices suffices in order to establish the main feature of the NNSD when integrability is broken: linear repulsion of levels with a slope which is inversely proportional to the small parameter that measures the departure from integrability. This is as universal a behaviour one can hope to get for near-integrable systems.

For Gaussian-distributed off-diagonal matrix elements Φ is an error function and $\phi(u) = \sqrt{(2/\pi)}\, e^{-u^2/2}$. Here the scaling parameter λ measures the magnitude of the off-diagonal matrix elements. Thus a small λ indicates a small perturbation to the levels. We find finally:

$$P(s) = \frac{s}{2\lambda}\sqrt{\frac{2}{\pi}} \int_0^{\pi/2} d\theta\, e^{-s\sin\theta}\, e^{-s^2\cos^2\theta/2\lambda^2}. \tag{7}$$

For small s we can neglect $e^{-s\sin\theta}$ and the angular integral can be readily computed:

$$P(s) = \frac{s}{\lambda}\sqrt{\frac{\pi}{8}}\, I_0(s^2/16\lambda^2)\, e^{-s^2/16\lambda^2} \qquad \text{for } s \ll 1 \tag{8}$$

where I_0 is a Bessel function, which tends to one as s goes to zero.

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The distribution obtained in (7) under integral form can be useful in another way also. When $s/\lambda \gg 1$ and $s \approx 1$ and we remark that the main contribution to the integral comes from the region where $\cos\theta \ll 1$ i.e. $\theta \approx \pi/2$. In this case we can replace $e^{-s\sin\theta}$ by e^{-s} and we obtain an approximate expression for the NNSD which reads:

$$P(s) = \frac{s}{\lambda}\sqrt{\frac{\pi}{8}}\, I_0(s^2/16\lambda^2)\, e^{-s^2/16\lambda^2}\, e^{-s}. \tag{9}$$

We remark that for $s \to \infty$ the term multiplying e^{-s} goes exactly to 1, which means that the distribution (9) goes over to the Poisson exponential. Thus we have another expression for the representation of the NNSD in the transition region, which, although derived through drastic approximations, presents the correct limits at $s \to 0$ and $s \to \infty$.

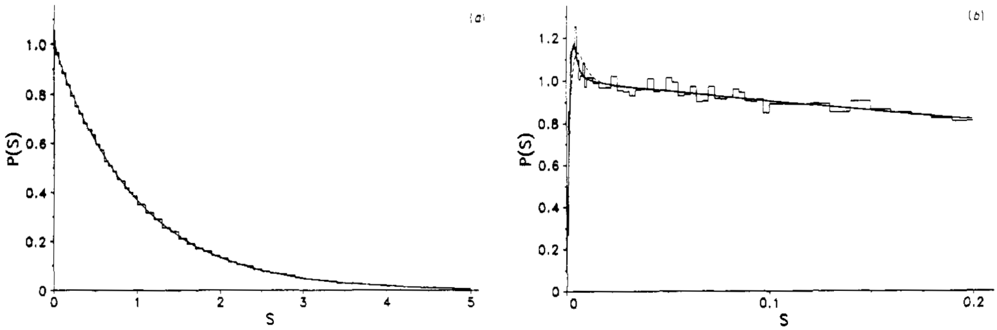

Figure 1. Nearest-neighbour spacings distribution obtained for various values of the (perturbation) parameter λ. $\lambda = 0.001$. Figures (*b*) give an enlarged view of the small-spacings' region. The continuous curve corresponds to a fit with expression (10), while the chain curve represents expression (11). The Poisson distribution is given in by the dot curve.

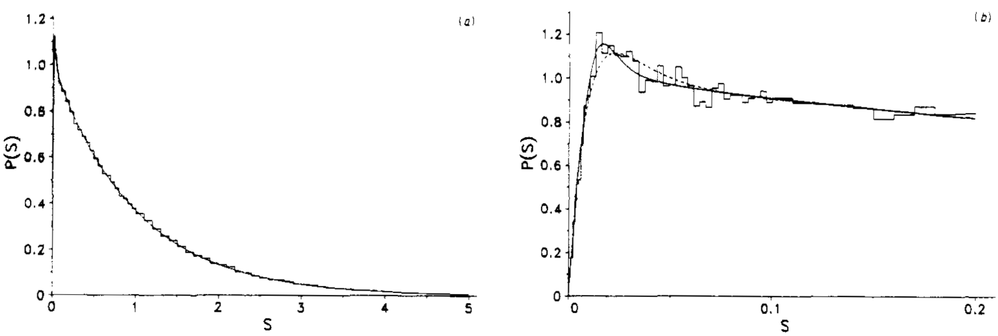

Figure 2. Same as figure 1 but $\lambda = 0.005$.

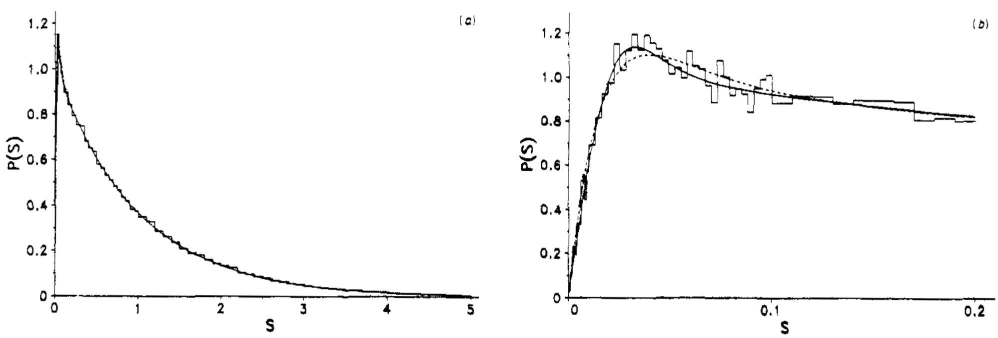

Figure 3. Same as figure 1 but $\lambda = 0.010$.

Still the fact that (9) is only approximate can be seen in the fact that $\int_0^\infty P(s)\,\mathrm{d}s \neq 1$ and $\int_0^\infty sP(s)\,\mathrm{d}s \neq 1$. However the deviation from 1 goes to zero at least as fast as λ and thus stays very small in the region near integrability. As we will see further on, this expression will be quite useful.

To our knowledge, very few studies of the pertinence of RM in the transitional region exist (but see the seminal work of Pandey [21]). A comparison of the structure of RM exhibiting Poisson and Wigner NNSD's suggests that for the transition region

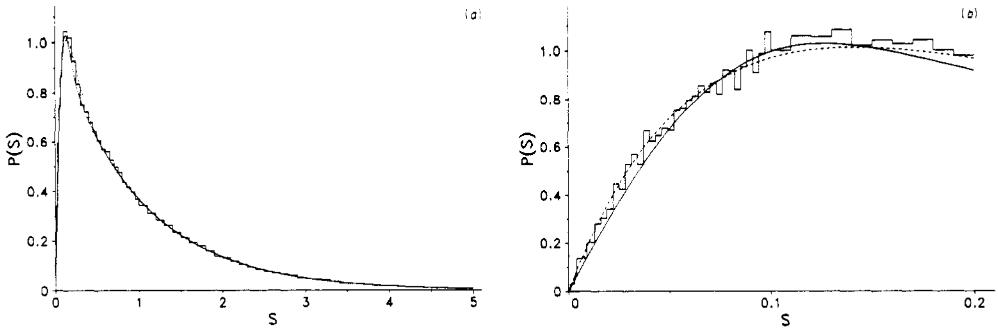

Figure 4. Same as figure 1 but $\lambda = 0.050$.

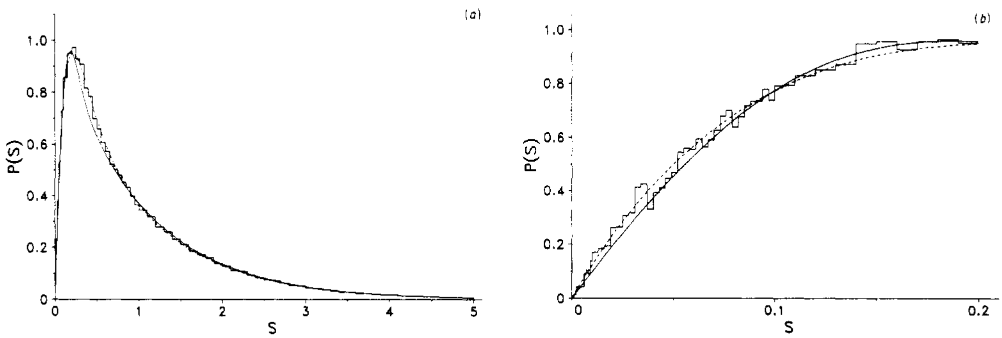

Figure 5. Same as figure 1 but $\lambda = 0.100$.

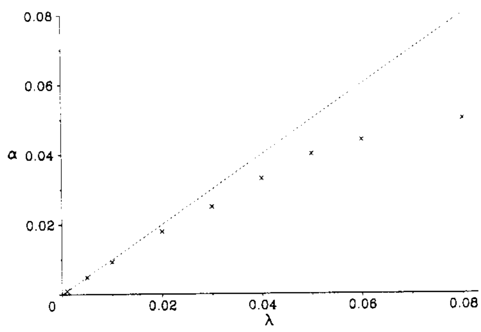

Figure 6. Best fitting parameter α, for expression (10), as a function of λ, together with the line $\alpha = \lambda$ (broken curve). Notice that for small enough λ's we obtain $\alpha \approx \lambda$.

one must consider band matrices [22], i.e. matrices where the non-zero matrix elements are concentrated around the diagonal. (The results of a recent study [23] on 3×3 band matrices do not apply to our case because of a different choice of the matrix element distribution and the fact that they are specific to the 3×3 case). In what follows we will consider tridiagonal matrices. The reason for this choice is mainly the fact that the diagonalisation of such matrices is extremely fast and accurate. Thus we can obtain high-quality statistics making possible a fine-scale study of the transition region. But a physical motivation also exists. In a recent work [24], Feingold *et al* have shown that, at the semi-classical limit, by an appropriate ordering of the eigenbasis of a given

Hamiltonian, a Hamiltonian resulting from the first through a perturbation has a banded structure. The choice of elements of our matrices is carried through in three steps. First the diagonal elements, for a $N\times N$ matrix, are randomly chosen in the interval $[0, N]$, with a constant probability. Next we reorder them in order of increasing magnitude on the diagonal. Then we choose the off-diagonal matrix elements according to the probability given by a Gaussian with variance λ, which will be the small parameter in our calculations. Only tridiagonal matrices have been considered. In fact, when λ is small, elements further away from the diagonal contribute higher powers of λ in the levels and are, thus, of negligible effect. This, of course, ceases to be true for larger λ, and the behaviour of a tridiagonal matrix at $\lambda\approx 1$ can be substantially different from that of a full matrix. The matrix size we have used throughout our calculations is $N=200$ and in order to improve the statistics several (more than 1000) such matrices were diagonalized. The NNSD of their spectra were superimposed in order to get the final distribution. The use of hundreds of thousands of levels allowed us to obtain very detailed results in the 'near-integrability' region of very small λ's even for extremely small spacings (figure 1).

We remark readily that the NNSD: (*a*) goes to zero as $s\to 0$ and (*b*) does so linearly. For a larger s the distribution $P(s)$ goes over to a pure Poisson e^{-s}. Since the integral over each of the two distributions must be equal to unity, it is clear that for spacings s of the order of λ the histogram must overshoot the Poisson exponential. This is indeed the case. In order to describe the global behaviour of $P(s)$ we have considered two parametrizations. The first is directly inspired from (9):

$$P(s)=\frac{s}{\alpha}\sqrt{\frac{\pi}{8}}\,I_0(s^2/16\alpha^2)\,\mathrm{e}^{-s^2/16\alpha^2}\,\mathrm{e}^{-s} \qquad (10)$$

where α is a free parameter. The second

$$P(s)=\mathrm{e}^{-s}\left[\left(1-\frac{1}{\beta\gamma}\right)+\frac{(\beta+1)^2}{\beta(\gamma-\beta)}\,\mathrm{e}^{-\beta s}-\frac{(\gamma+1)^2}{\gamma(\gamma-\beta)}\,\mathrm{e}^{-\gamma s}\right] \qquad (11)$$

has been constructed so as to have the required properties: $P(s)\propto s$ as $s\to 0$, $P(s)\approx \mathrm{e}^{-s}$ as $s\gg\lambda$, and moreover satisfying $\int_0^\infty P(s)\,\mathrm{d}s=1$ and $\int_0^\infty sP(s)\,\mathrm{d}s=1$. As we can see in figures 1-5 both expressions represent the data in a most satisfying manner (once the α, β, γ have been adjusted through a χ^2 fit). In particular the (quite appreciable) overshoot around $s\approx\lambda$ (for very small values of λ) is very well reproduced.

As we have stated previoisly, the transition region near integrability does not present a universal behaviour except from the fact that $P(s)$ vanishes at $s\to 0$ with a slope inversely proportional to the small parameter characterizing the off-diagonal perturbation. In order to check this feature we have estimated the slope of $P(s)$ at the origin using (10), obtaining the parameter α. In figure 6 we present the dependence of α, obtained through a χ^2 fit, as a function of the small parameter λ. We remark that as $\lambda\to 0$, α goes over to λ, in perfect agreement with the predictions.

Thus the study of RM, with the appropriate structure, has allowed us to obtain detailed statistics in the transition region 'near integrability'. The latter term is to be understood, as explained at the outset, through the relation of the statistics of RM eigenvalues and those of the levels of quantum Hamiltonians. Thus, at least for the NNSD, our results can be considered as describing also the behaviour of the quantal spectra of near-integrable Hamiltonians. Moreover the two parametrizations (10) and (11) that we have proposed can be particularly useful in the description of the distributions obtained through realistic calculations in the transition region.

Note added in proof. After this paper was completed and submitted for publication, we came across the paper by F Leyvraz and T H Seligman entitled 'Self-consistent perturbation theory for random matrix ensembles' (1990 *J. Phys. A: Math. Gen.* **18** 1555). They treat a problem very similar to ours and derive, among others, an approximate expression for the two-point correlation function ρ_2 of levels in the case of a Poisson ensemble subject to a GOE perturbation. The small spacing behaviour of ρ_2 is in perfect agreement with the NNSD obtained in our approach. Incidentally, ρ_2 given as an integral in their equation (4.2) can be expressed in closed form. For $C=2$, we find $\rho_2(x;\lambda)=\sqrt{2\pi}\xi\,\mathrm{e}^{-\xi^2}I_0(\xi)$ where $\xi=x/4\sqrt{\lambda}$ and I_0 is a Bessel function.

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